Chapter 9

Counting

9.1 Fundamental principles

The subject of *enumerative combinatorics* is counting. Generally, there is some set A and we wish to calculate the size $|A|$ of A. Here are some sample problems:

- How many ways are there to seat n couples at a round table, such that each couple sits together?
- How many ways are there to express a positive integer n as a sum of positive integers?

There are a number of basic principles that we can use to solve such problems.

The sum principle: Consider n sets A_i, for $1 \leq i \leq n$, that are *pairwise disjoint*, namely $A_i \cap A_j = \emptyset$ for all $i \neq j$. Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| .$$

For example, if there are n ways to pick an object from the first pile and m ways to pick on object from the second pile, there are $n+m$ ways to pick an object altogether.

The product principle: If we need to do n things one after the other, and there are c_1 ways to do the first, c_2 ways to do the second, and so on, the number of possible courses of action is $\prod_{i=1}^{n} c_i$. For example, the number of possible three-letter words in which a letter appears at most once that can be constructed using the English alphabet is $26 \cdot 25 \cdot 24$: There are 26 possibilities for the first letter, then 25 possibilities for the second, and finally 24 possibilities for the third.

The bijection principle: As we have seen, there exists a bijection from A to B if and only if the size of A equals the size of B. Thus, one way to count the number of elements in a set A is to show that there is a bijection from A to some other set B

and to count the number of elements in B. Often there is no need to explicitly specify the bijection and prove that it is such: At this point in the course, you can omit some low-level details from the written proofs in your homework solutions, *as long as you are certain that you could reproduce these details if asked to do so.* For example, you can simply state and use the observation that the number of ways to seat n people in a row is the same as the number of ways to order the integers $1, 2, \ldots, n$, which is the same as the number of n-element sequences that can be constructed from the integers $1, 2, \ldots, n$ (without repetition), which is the same as the number of bijections $f : A \to A$, for $A = \{1, 2, \ldots, n\}$. You should always make sure that you yourself fully understand why such equalities hold whenever you use them! Obviously, if you don't, you'll end up relying on equalities that are simply not true, which is not such a great idea. If in doubt, write down a complete proof to make sure your reasoning is correct.

9.2 Basic counting problems

Choosing an ordered sequence of distinct objects with repetition. How many ways are there to pick an ordered sequence of k objects from a pool with n types of objects, when repetitions are allowed? (That is, we can pick an object of the same type more than once.) Well, by the product principle, there are n options for the first object, n options for the second, and so on. Overall we get n^k possible sequences. What follows is a somewhat more formal argument by induction. Observe that the number of sequences as above is the same as the number of functions from a set of k elements to a set of n elements. (Make sure you understand this.)

Theorem 9.2.1. *Given sets A and B, such that $|A| = k$ and $|B| = n$, the number of functions $f : A \to B$ is n^k.*

Proof. Induction on k. If $k = 0$ the set A has no elements and there is only one mapping from A to B, the empty mapping. (Recall that a function $f : A \to B$ is a subset of $A \times B$, and if $A = \emptyset$ then $A \times B = \emptyset$.) We suppose the claim holds for $|A| = m$ and treat the case $|A| = m + 1$. Consider some element $a \in A$. To specify a function $f : A \to B$ we can specify $f(a) \in B$ and a mapping $f' : A \setminus \{a\} \to B$. There are n possible values of $f(a) \in B$, and for each of these there are n^m mappings f' by the induction hypothesis. This results in n^{m+1} mappings f and completes the proof by induction. □

Choosing an ordered sequence of distinct objects *without* repetition. How many ways are there to pick an ordered sequence of k objects from a set of n objects when only one copy of each object is available, so there can be no repetitions? Again we can use the product principle. Observe that the first object in the sequence can be chosen from n distinct objects. Once the first one is picked, there are only $n - 1$ possibilities for the second object. After that there are $n - 2$ objects to choose from, and so on. Overall we get that the desired quantity is

$$n(n-1)\cdots(n-k+1)=\prod_{i=0}^{k-1}(n-i).$$

This is called a *falling factorial* and denoted by $(n)_k$ or $n^{\underline{k}}$. We again provide a more formal proof by induction, observing that the number of ways to pick an ordered sequence of k objects from a collection of n distinct ones without replacement is equal to the number of *one-to-one* functions $f : A \to B$, where $|A| = k$ and $|B| = n$.

Theorem 9.2.2. *Given sets A and B, such that $|A| = k$ and $|B| = n$, the number of* one-to-one *functions $f : A \to B$ is $(n)_k$.*

Proof. Induction on k. When $|A| = 0$, there is one mapping f as described, the empty mapping, and $(n)_k$ is the empty product, equal to 1. Suppose the claim holds for $|A| = m$ and consider the case $|A| = m + 1$. Fix an element $a \in A$. To specify f we specify $f(a)$ and a mapping $f' : A \setminus \{a\} \to B$. There are n possible values for $f(a) \in B$. Consider a specific such value $f(a) = b$. Since f is one-to-one, no element of $A \setminus \{a\}$ can be mapped to b. Thus f' has to be a one-to-one-mapping from $A \setminus \{a\}$ to $B \setminus \{b\}$. By the induction hypothesis, the number of such mappings is $(n-1)_m$. The number of possible mappings f is thus $n \cdot (n-1)_m = (n)_{m+1}$. □

Permutations. How many ways are there to arrange n people in a row? How many ordered n-tuples are there of integers from the set $\{1, 2, \ldots, n\}$? Haw many distinct rearrangements are there of the integers $1, 2, \ldots, n$? How many bijections are there from the set $\{1, 2, \ldots, n\}$ to itself? The answer to these questions is the same, and follows from Theorem 9.2.2. A bijection from a set A to itself is called a *permutation* of A. The number of permutations of the set $\{1, 2, \ldots, n\}$ is precisely the number of one-to-one functions from this set to itself, and this number is $(n)_n = n\cdot(n-1)\cdots 2\cdot 1$. This quantity is called "n factorial" and is denoted by $n!$. We can now observe that

$$(n)_k = \frac{n!}{(n-k)!}.$$

It is important to remember that $0! = 1$, since $0!$ is the empty product. Here is a list of values of $n!$ for $0 \le n \le 10$:

$$1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800$$

Seating at a round table. We've arranged n people in a row, now it's time to sit them down. So how many ways are there to seat n people at a round table? Let's be precise about what we mean: Two seating arrangements are considered identical if every person has the same neighbor to her right. In other words, rotations around the table do not matter. Here is how this problem can be tackled: Fix one person a and sit her down anywhere. This now fixes $n - 1$ possible positions for the others: "first person to the right of a", "second person to the right of a", and so on until "$(n-1)$-st person to the right of a". The number of ways to arrange the others in these $n - 1$ positions is $(n-1)!$, which is also the answer to the original question.

Choosing an *unordered* collection of distinct objects *without* repetition. How many ways are there to pick a *set* of k objects from a set of n objects? Since we are picking a set, we do not care about order, and there are no repetitions. Notice that every such set can be ordered in $k!$ ways. That is, each set corresponds to $k!$ distinct ordered k-tuples. Now, we know that the number of ordered k-tuples that can be picked from a collection of n distinct objects is $(n)_k$. Thus if we denote by X the number of sets of cardinality k that can be picked from a collection of n distinct objects, we get

$$\begin{aligned} X \cdot k! &= (n)_k \\ X &= \frac{(n)_k}{k!} \\ X &= \frac{n!}{k!(n-k)!}. \end{aligned}$$

This quantity X is denoted by $\binom{n}{k}$, read "n choose k". This is such an important quantity that we emphasize it again: The number of k-element subsets of an n-element set is $\binom{n}{k}$, defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}.$$

We can see that $\binom{n}{0} = \binom{n}{n} = 1$, and we define $\binom{n}{k} = 0$ when $k > n$ or $k < 0$.

The number of subsets. We have seen that the number of k-element subsets of an n-element set is $\binom{n}{k}$. How many subsets of an n-element set are there overall, of any size? Yes, it is time to prove the neat formula we've been using all along:

Theorem 9.2.3. *For a set A,*

$$\left|2^A\right| = 2^{|A|}.$$

Proof. By induction. When $|A| = 0$, $A = \emptyset$. Hence, A has only one subset (itself) and the formula holds since $2^0 = 1$. Assume the formula holds when $|A| = k$ and consider the case $|A| = k + 1$. Fix an element $a \in A$. A subset of A either contains a or not. The subsets of A that do not contain a are simply subsets of $A \setminus \{a\}$ and their number is 2^k by the induction hypothesis. On the other hand, each subset of A that does contain a is of the form $\{a\} \cup X$, for $X \subseteq A \setminus \{a\}$. Thus there is a bijective mapping between subsets of A that contain a and subsets of $A \setminus \{a\}$. The number of such subsets is again 2^k. Overall we get that the number of subsets of A is $2^k + 2^k = 2^{k+1}$, which completes the proof by induction. □

Here is another instructive way to prove Theorem 9.2.3: Consider the set of functions $f : A \to \{0, 1\}$. These functions assign a value of 0 or 1 to every element of A. In this way, such a function f uniquely specifies a subset of A. Namely, the elements x for which $f(x) = 1$ are the elements that belong to the subset of A specified by f. This defines a bijection between such functions f and subsets of A. By Theorem 9.2.1, the number of functions f from A to $\{0, 1\}$ is $2^{|A|}$, which proves Theorem 9.2.3.

We can use Theorem 9.2.3 to derive an interesting identity. We now know that the overall number of subsets of an n-element set is 2^n. Previously we have seen that the number of k-element subsets of an n-element set is $\binom{n}{k}$. By the sum principle, we get

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Choosing an *unordered* collection of distinct objects *with* repetition. How many ways are there to pick a collection of k objects from a pool with n types of objects, when repetitions are allowed? We can reason as follows: The number of ways to pick k objects from a pool with n types of objects is the same as the number of ways to put k balls into n bins. Imagine these bins aligned in a row. A "configuration" of k balls in n bins can be specified as a sequence of $n-1$ "|" symbols and k "*" symbols, as in

$$* * \,|\,|\, * \,|\, * * * \,|$$

This sequence encodes the configuration where $k = 6$ and $n = 5$, and there are two balls in bin number 1, one ball in bin number 3, and three balls in bin number 4. How many such configurations are there? A configuration is uniquely specified by the positions of the k "*" symbols. Thus specifying a configuration amounts to choosing which of the $n+k-1$ symbols are going to be "*". This simply means we need to choose a k-element subset from a set of size $n+k-1$. The number of ways to pick a collection of k objects from a pool of n types of objects with repetitions is thus

$$\binom{n+k-1}{k}.$$